Calculus Exercises: Solutions

split into factors:
(1.) $x^2 + 5x + 6$

Assume the factors to be $(x + a)$ and $(x + b)$:

$$(x + a)(x + b) =$$
$$x^2 + ax + bx + ab =$$
$$x^2 + (a + b)x + ab$$

if this equates $x^2 + 5x + 6$, then $a + b = 5$ and $ab = 6$, so a possible solution is $a = 2$ and $b = 3$, *i.e.*

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

Alternative: use the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (with a, b, and c the coefficients of the quadratic equation) to find the values of x where the equation equals 0. This gives $x = -2$ and $x = -3$ which leads to the same solution.

(2.) $x^2 - x - 6$

Similar to **(1.)**: $x^2 - x - 6 = (x - 3)(x + 2)$.

(3.) $x^2 + x - 20 = (x + 5)(x - 4)$
(4.) $3x^2 - 24x + 45$

There are several correct solutions. One is to divide by the coefficient of x^2 (here 3) to get a form similar to the one in the previous questions:

$$3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x - 3)(x - 5).$$

(5.) $2x^2 + 3x - 2 = 2(x^2 + \frac{3}{2}x - 1) = 2(x - \frac{1}{2})(x + 2)$

(6.) Compute $\dfrac{100!}{97!}$

$$\frac{100!}{97!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \ldots 1}{97 \cdot 96 \cdot 95 \ldots 1} = 100 \cdot 99 \cdot 98 = 970200.$$

(7.) Simplify (for n even): $\dfrac{n!(n-2)!(n-4)!\dots 0!}{(n-1)!(n-3)!\dots 1!}$

$$\frac{n!(n-2)!(n-4)!\dots 0!}{(n-1)!(n-3)!\dots 1!} = \frac{n!}{(n-1)!}\cdot\frac{(n-2)!}{(n-3)!}\cdot\frac{(n-4)!}{(n-5)!}\dots\frac{2!}{1!}\cdot 0! = n(n-2)(n-4)\dots 2\cdot 1.$$

(8.) Simplify $\sqrt[4]{(2^{1/3})^6(\sqrt{2})^{12}}$

$$\sqrt[4]{(2^{1/3})^6(\sqrt{2})^{12}} = \sqrt[4]{2^2\cdot 2^6} = \sqrt[4]{2^8} = 2^{8/4} = 2^2 = 4.$$

(9.) Simplify $\exp(-\ln 4 - \ln 3)$

$$\exp(-\ln 4 - \ln 3) = \exp\left(\ln\frac{1}{4} - \ln 3\right) = \exp\left(\ln\frac{1/4}{3}\right) = \frac{1/4}{3} = \frac{1}{12}.$$

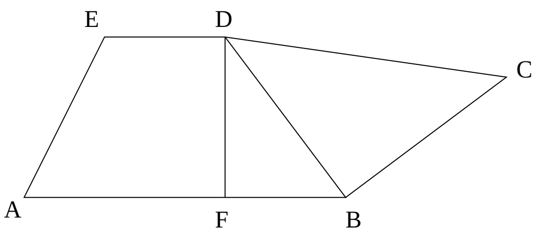
(10.) Find the center coordinates and radius of the circle $x^2+y^2-4x+6y=12$

We need to work towards the form $(x-a)^2+(y-b)^2=r^2$. Our equation contains $x^2-4x\dots$ which suggests using $(x-2)^2$. But using $(x-2)^2 = x^2-4x+4$ adds $\dots+4$, so we must add 4 to the right side as well. A similar reasoning is followed for the y:

$$\begin{aligned}
&x^2+y^2-4x+6y=12\\
&(x^2-4x)+(y^2+6y)=12\\
&(x^2-4x+4)+(y^2+6y+9)=12+4+9\\
&(x-2)^2+(y+3)^2=25.
\end{aligned}$$

This corresponds to a circle with center $(2,-3)$ and radius $\sqrt{25}=5$.

(11.) Given this figure and data



$AB \parallel DE$
$AB = 8$
$DE = 3$
$DF = 4$
$AF = 5$
$\angle B$ (right) $= \angle F = 90^o$
$BC = BD$,

compute the area $ABCDE$.

The area of the trapezoid $ABDE$ equals $\frac{8+3}{2}\cdot 4 = 22$. We only need to compute the area of BCD now. Using the pythagorean theorem: $BD^2 = BF^2 + DF^2$ we find $BD = 5$ and hence

$BC = 5$. Because $\angle B = 90^o$, BD and BC are the base and height of triangle BCD. So the area of BCD equals $\frac{1}{2} \cdot 5 \cdot 5 = 12\frac{1}{2}$. Area $ABCDE$ is therefore $22 + 12\frac{1}{2} = 34\frac{1}{2}$.

(12.) Compute the area of the triangle formed by the intersections of the lines:

$$\begin{array}{rrcl} l_1: & y & = & 1+x \\ l_2: & y & = & 5-x \\ l_3: & y & = & 3-\frac{1}{2}x. \end{array}$$

The three intersection points A, B, and C are found by equating l_1 & l_2, l_2 & l_3, and l_1 & l_3 respectively. for example A: $1 + x = 5 - x$, which leads to $x = 2$. Substituting in l_1 (or l_2) leads to $y = 3$, so $A = (2, 3)$. Similarly, we find $B = (4, 1)$ and $C = (4/3, 7/3)$.

Call the triangle sides opposite to A, B, and C repectively a, b, and c. By using the pythagorean theorem on the triangle vertices we find:

$$\begin{array}{l} a^2 = (4-4/3)^2 + (1-7/3)^2 = 80/9 \\ b^2 = (2-4/3)^2 + (3-7/3)^2 = 8/9 \\ c^2 = (2-4)^2 + (3-1)^2 = 8 \end{array}$$

There are several ways to proceed now:
Alternative 1: Observe that $l_1 \perp l_2$. (You can see this because the products of their slopes are -1, or, alternatively, because the inner product of the direction vectors of the lines equals 0.) This means that c and b are base and height of the triangle, and hence the area equals $\frac{1}{2}cb = \frac{1}{2}\sqrt{8}\sqrt{8/9} = 4/3$
Alternative 2: From linear algebra, we know the area of the triangle can be expressed in terms of the determinant of the matrix formed by the vectors corresponding to two of the triangle sides:

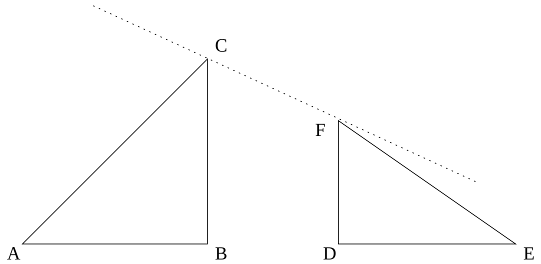
$$\frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -8/3 & -2 \\ 4/3 & 2 \end{vmatrix} = \frac{1}{2}\left(-\frac{16}{3} + \frac{8}{3}\right) = -\frac{4}{3}.$$

Taking the absolute value –the determinant is signed, but area is of course always positive– gives us $\frac{4}{3}$ again.

Alternative 3: Using only calculus (and not observing the perpendicularity in alternative 1), you can still solve this 'brute force' by computing a height of the triangle. For example: call $a = BC$ the base of the triangle. Call h the height of the triangle; h goes through A and is perpendicular to a. Call γ the angle at C.
To compute $h = b\sin\gamma$, we need to know γ. We can compute γ from the cosine law: $c^2 = a^2 + b^2 - 2ab\cos\gamma$. Substituting a, b, and c and solving gives $\cos\gamma = 1/\sqrt{10} \approx 0.316$. Using a calculator we find $\gamma \approx 1.2491$ and $\sin\gamma \approx 0.9487$ (You can also do this without approximations

by using the identity $\cos^2\gamma + \sin^2\gamma = 1$.) Hence, $h \approx 0.8944$. The triangle area is then $\frac{1}{2}ah \approx 1.3333$.

(13.) Give the equation of the line CF given



$A = (0, 0)$
$E = (16, 0)$
$ABDE \parallel x$-axis
$\angle A = \pi/4$
$\angle B = \angle D = \pi/2$
$\angle E = \pi/6$
$BC = 6$
$DF = 4$.

Their are many possible solutions here. This is one:
Solve AB from $\tan\frac{\pi}{4} = \frac{6}{AB}$. This gives $AB = 6$.
Solve DE from $\tan\frac{\pi}{6} = \frac{4}{DE}$. This gives $DE = \frac{12}{\sqrt{3}} = 4\sqrt{3}$.
So $C = (6, 6)$ and $F = (16 - 4\sqrt{3}, 4)$.
Assume the line through CF equals $y = ax + b$. Filling in the coordinates of C and F leads to the equations

$$\begin{aligned} 6 &= 6a + b \\ 4 &= (16 - 4\sqrt{3})a + b \end{aligned}$$

The quickest way to solve this is to eliminate b by subtracting the two equations:
$2 = a(6 - 16 + 4\sqrt{3})$, which leads to $a = \frac{1}{2\sqrt{3}-5}(\approx -0.651)$. Substituting this in one of the original equations leads to $b = 6 - \frac{6}{2\sqrt{3}-5}(\approx 9.907)$.

Compute the derivative of
(14.) $\sin(x) + \cos(x)$

The derivative of a sum equals the sum of the derivatives, so

$$\frac{d}{dx}(\sin(x) + \cos(x)) = \cos(x) - \sin(x).$$

(15.) $\sin(x)\cos(x)$

Using the product rule:

$$\frac{d}{dx}(\sin(x)\cos(x)) = \cos(x)\cos(x) + \sin(x)(-)\sin(x) = \cos^2(x) - \sin^2(x) = \cos(2x).$$

(16.) $\sin(\cos(x))$

By the chain rule with $u = \sin$ and $v = \cos$:

$$\frac{d}{dx}(\sin(\cos(x))) = \cos(\cos(x))(-)\sin(x) = -\sin(x)\cos(\cos(x)).$$

Compute the extrema of
(17.) $f(x) = (x-2)(x-4)$

$f(x) = (x-2)(x-4) = x^2 - 6x + 8$, so $f'(x) = 2x - 6$, so $f'(x) = 0$ if $x = 3$. The extremum is therefore $(3, f(3)) = (3, -1)$.

(18.) $f(x) = \sin(x)\cos(x)$, with $x \in [-\pi, \pi]$

$f'(x) = \cos(2x)$ (see ex.15).

$$\begin{aligned} &f'(x) = 0 \\ &\cos(2x) = 0 \\ &2x = \tfrac{\pi}{2} + k\pi \quad \text{for integer } k \\ &x = \tfrac{\pi}{4} + k\tfrac{1}{2}\pi. \end{aligned}$$

Solutions within the domain are $\{-\frac{3}{4}\pi, -\frac{1}{4}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi\}$. The corresponding extrema are $(-\frac{3}{4}\pi, \frac{1}{2}), (-\frac{1}{4}\pi, -\frac{1}{2}), (\frac{1}{4}\pi, \frac{1}{2}), (\frac{3}{4}\pi, -\frac{1}{2})$.

(19.) What is the maximum area of a rectangle with circumference 8 and width x?

Call the rectangle width x and the height y. The circumference equals $2x+2y = 8$, so $y = 4-x$. The rectangle area equals $xy = x(4-x) = 4x - x^2$. This area is maximum if the derivative is zero, so when $4 - 2x = 0$, so when $x = 2$ (so the rectangle is in fact a square). The area then is $2 \cdot 2 = 4$.